# The Inflation of Thick-Walled Inner Tubes and Tires

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## Synopsis

Approximate analytic results for two problems involving the inflation of rubber inner tubes and tires are summarized with a view to prompting a comparison of the theory with experimental results. The problems considered are the uniform inflation by internal pressure of a toroidal rubber inner tube or tire which is a perfect torus either in the deformed or undeformed state. The inner tubes or tires are assumed to be free to inflate, that is, they are assumed not to be constrained, for example by a wheel.

## INTRODUCTION

The inflation by uniform internal pressure of toroidal rubber inner tubes and tires is a technological problem of continuing importance. Kydoniefs and Spencer[1] give a perturbation solution to the inflation problem of a thick-walled perfectly elastic material which is a torus in its deformed state. Their solution is approximate in the sense that the radius of the larger circle which generates the torus is assumed to be small in comparison to the overall radius of the torus. With this assumption these authors obtained the zero- and first-order terms of the solution, the latter being calculated only for the special case of the neo-Hookean material. Recently the author[2] has extended the analysis of Kydoniefs and Spencer[1] to the general perfectly elastic material, as well as giving a similar approximate solution to the uniform inflation of a thick-walled inner tube or tire which is a torus in its undeformed state. As far as practical purposes are concerned, the nature of these approximations is such that the simple zero-order contribution could well be adequate to determine the main characteristics of pressure-deformation curves. This would have to be confirmed experimentally and the purpose of this article is to present a concise account of the basic equations derived in Hill[2] and Kydoniefs and Spencer[1] for the zero-order solution, with a view to prompting experimental verification of these formulae.

The solution of both problems are approximated by the known solution to the problem of the uniform inflation by internal pressure of a long circular cylindrical tube due originally to Rivlin.[3] This deformation [eq. (16)] involves two arbitrary constants $K$ and $\gamma$. For both problems these two constants are determined from the pressure boundary conditions and by considering the overall equilibrium of a sector of the torus. The two basic equations for the uniform inflation of a thick-walled perfectly incompressible elastic material which is a torus in its undeformed state are (21) and (23) where the response functions $\phi_1$ and $\phi_2$ are defined by (6) and the subscript zero denotes evaluation at the zero-order deformation given by (16). Similarly for uniform inflation to a torus in its deformed state the two fundamental equations are (29) and (32) and in this case the response functions are defined by (11). Almost certainly the zero-order approx-

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imations will adequately describe the behavior of bicycle inner tubes and tires. However, it remains to be confirmed experimentally whether these simple approximations are applicable to inner tubes and tires of the dimensions used on motor vehicles.

In the following section some theoretical preliminaries are noted. In the section thereafter the inflation from a torus in its undeformed state is considered while in the final section the problem of the inflation to a torus in its deformed state is examined. Numerical results are given for both problems for the Mooney material.

## COORDINATES AND TERMINOLOGY

In the undeformed body with rectangular cartesian coordinates $(X,Y,Z)$ we use toroidal coordinates $(R,\Theta,\Phi)$ defined by

$$\begin{aligned} R &= \{[(X^2+Y^2)^{1/2}-b]^2+Z^2\}^{1/2} \\ \Theta &= \tan^{-1}(Y/X) \\ \Phi &= \tan^{-1}\{Z/[(X^2+Y^2)^{1/2}-b]\} \end{aligned} \tag{1}$$

where $b$ is a constant which denotes the distance from the origin to the center of the concentric circles which generate the torus (see Fig. 1). In the deformed body with rectangular cartesian coordinates $(x,y,z)$ we use $(r,\theta,\phi)$ given by

$$\begin{aligned} r &= \{[(x^2+y^2)^{1/2}-c]^2+z^2\}^{1/2} \\ \theta &= \tan^{-1}(y/x) \\ \phi &= \tan^{-1}\{z/[(x^2+y^2)^{1/2}-c]\} \end{aligned} \tag{2}$$

with $c$ a constant (see Fig. 2).

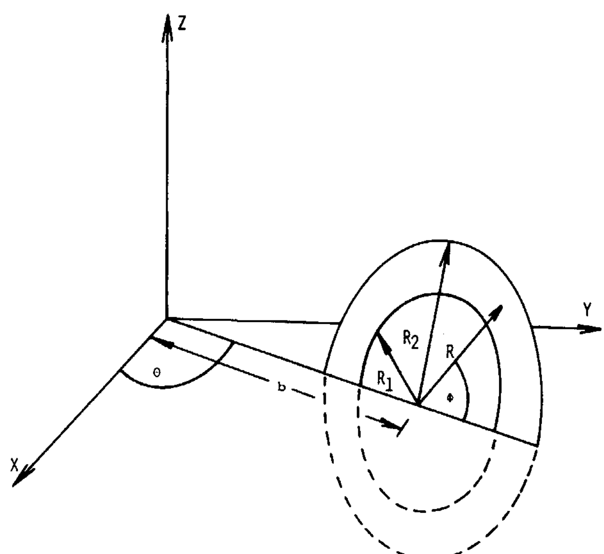


Fig. 1. Coordinates for the undeformed configuration.

In the following section we consider a torus which in its undeformed state is obtained by rotating the region between two concentric circles of radii $R_1$ and $R_2$ about the $Z$ axis. If the torus is inflated by a uniform internal pressure $P_1$, we suppose an axially symmetric deformation of the form

$$r = r(R,\Phi), \quad \theta = \Theta, \quad \phi = \phi(R,\Phi) \tag{3}$$

and we use the coordinates in the undeformed body as the independent variables. Here the constant $b$ in (1) is prescribed while $c$ has yet to be determined. We consider a homogeneous isotropic incompressible hyperelastic material with general strain–energy function $\Sigma(I_1,I_2)$, where $I_1$ and $I_2$ are the principal invariants of the finite deformation strain tensor which are given by

$$I_1 = I + \alpha^2, \quad I_2 = \alpha^2 I + \alpha^{-2} \tag{4}$$

where $I$ and $\alpha$ are defined by

$$I = \left(\frac{\partial r}{\partial R}\right)^2 + \frac{1}{R^2}\left(\frac{\partial r}{\partial \Phi}\right)^2 + r^2\left[\left(\frac{\partial \phi}{\partial R}\right)^2 + \frac{1}{R^2}\left(\frac{\partial \phi}{\partial \Phi}\right)^2\right], \quad \alpha = \left(\frac{c + r\cos\phi}{b + R\cos\Phi}\right). \tag{5}$$

We shall need the following response functions:

$$\phi_1 = 2\frac{\partial \Sigma}{\partial I_1} + 2\alpha^2\frac{\partial \Sigma}{\partial I_2} = 2\frac{\partial \Sigma}{\partial I}$$

$$\phi_2 = 2\frac{\partial \Sigma}{\partial I_1} + 2\left(I - \frac{1}{\alpha^4}\right)\frac{\partial \Sigma}{\partial I_2} = 2\frac{\partial \Sigma}{\partial \alpha^2} \tag{6}$$

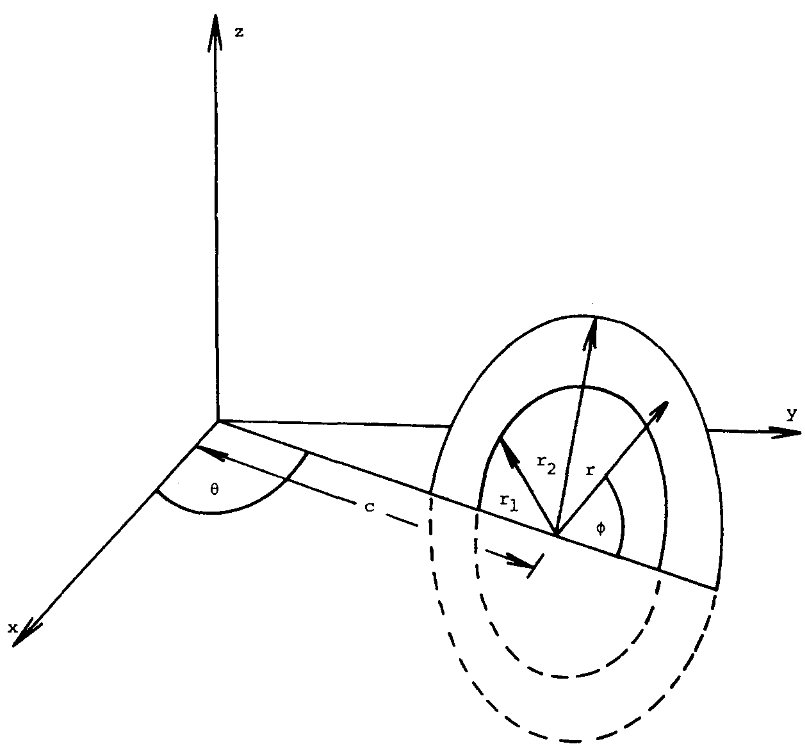


Fig. 2. Coordinates for the deformed configuration.

We remark that the incompressibility condition for (3) is

$$\frac{\partial r}{\partial R}\frac{\partial \phi}{\partial \Phi} - \frac{\partial r}{\partial \Phi}\frac{\partial \phi}{\partial R} = \frac{R(b + R\cos\Phi)}{r(c + r\cos\phi)} \tag{7}$$

In the final section we consider a torus which in its deformed state is obtained by rotating the region between two concentric circles of radii $r_1$ and $r_2$ about the $z$ axis. If the torus is held in this state by a uniform internal pressure $P_2$, then we use the coordinates in the deformed body as the independent variables, that is, in place of (3) we have

$$R = R(r,\phi), \quad \Theta = \theta, \quad \Phi = \Phi(r,\phi) \tag{8}$$

For this problem the constant $c$ in (2) is prescribed while $b$ has yet to be determined. The principal invariants become

$$I_1 = \beta^2 J + \beta^{-2}, \quad I_2 = J + \beta^2 \tag{9}$$

where $J$ and $\beta$ are defined by

$$J = \left(\frac{\partial R}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial R}{\partial \phi}\right)^2 + R^2\left[\left(\frac{\partial \Phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \Phi}{\partial \phi}\right)^2\right], \quad \beta = \left(\frac{b + R\cos\Phi}{c + r\cos\phi}\right) \tag{10}$$

We find that in this case the appropriate response functions are

$$\begin{aligned} \psi_1 &= 2\beta^2\frac{\partial \Sigma}{\partial I_1} + 2\frac{\partial \Sigma}{\partial I_2} = 2\frac{\partial \Sigma}{\partial J} \\ \psi_2 &= 2\left(J - \frac{1}{\beta^4}\right)\frac{\partial \Sigma}{\partial I_1} + 2\frac{\partial \Sigma}{\partial I_2} = 2\frac{\partial \Sigma}{\partial \beta^2} \end{aligned} \tag{11}$$

while the incompressibility condition for (8) becomes

$$\frac{\partial R}{\partial r}\frac{\partial \Phi}{\partial \phi} - \frac{\partial R}{\partial \phi}\frac{\partial \Phi}{\partial r} = \frac{r(c + r\cos\phi)}{R(b + R\cos\Phi)} \tag{12}$$

Finally in this section we note that the Mooney material has strain–energy function given by

$$\Sigma = C_1(I_1 - 3) + C_2(I_2 - 3) \tag{13}$$

where $C_1$ and $C_2$ are material constants. We use the usual notation $\Gamma = C_2/C_1$.

## UNIFORM INFLATION FROM A TORUS

We suppose that a torus which in its undeformed state is obtained by rotating the region between two concentric circles of radii $R_1$ and $R_2$ about the $Z$ axis is such that $\epsilon = R_2/b$ is small compared with unity. We introduce

$$\xi = \frac{R}{R_2}, \quad \eta = \frac{r}{R_2}, \quad \lambda = \frac{R_1}{R_2}, \quad \gamma = \frac{c}{b} \tag{14}$$

so that $\lambda \leqslant \xi \leqslant 1$ and the deformation (3) becomes

$$\eta = \eta(\xi,\Phi), \quad \theta = \Theta, \quad \phi = \phi(\xi,\Phi) \tag{15}$$

Now, in the limit of $\epsilon$ tending to zero, the problem of determining (15) reduces to the problem of the uniform inflation by internal pressure of a long circular cylindrical tube (see Rivlin[3]). Accordingly, we approximate (15) by

$$\eta = \gamma^{-1/2}(\xi^2 + K)^{1/2} + O(\epsilon), \quad \theta = \Theta, \quad \phi = \Phi + O(\epsilon) \tag{16}$$

where $\gamma$ and $K$ are two constants which are determined by the pressure boundary conditions and by considering the equilibrium of a sector cut off by the planes $\theta = \pm\theta_0$ and resolving forces in the $x$ direction. From (5) and (16) we find that the zero-order contributions to $I$ and $\alpha$ are given by

$$I_0 = \frac{1}{\gamma}\left[\frac{\xi^2}{\xi^2 + K} + \frac{\xi^2 + K}{\xi^2}\right], \quad \alpha_0 = \gamma \tag{17}$$

The physical components of the stress tensor can be shown to be given by

$$\begin{aligned}
\sigma_{rr} &= \frac{1}{\gamma}\left[-p_0(\xi) + \frac{\xi^2}{\xi^2 + K}\phi_{10}(\xi)\right] + O(\epsilon) \\
c^2\sigma_{\theta\theta} &= \frac{1}{\gamma}\left[-p_0(\xi) + \gamma^3\phi_{20}(\xi)\right] + O(\epsilon) \\
\sigma_{\phi\phi} &= \frac{1}{\gamma}\left[-p_0(\xi) + \frac{\xi^2 + K}{\xi^2}\phi_{20}(\xi)\right] + O(\epsilon) \\
\sigma_{r\phi} &= O(\epsilon)
\end{aligned} \tag{18}$$

while all other components are zero and $p_0(\xi)$ is given by

$$p_0(\xi) = \frac{\xi^2}{\xi^2 + K}\phi_{10}(\xi) - \int_\lambda^\xi \frac{K(2t^2 + K)\phi_{10}(t)}{t(t^2 + K)^2}\,dt + \pi_0 \tag{19}$$

where $\pi_0$ is a constant. From the boundary conditions of pressure $P_1$ at $R = R_1$ and external pressure zero at $R = R_2$, we have

$$\sigma_{rr} = -P_1 \quad \text{at } \xi = \lambda, \quad \sigma_{rr} = 0 \quad \text{at } \xi = 1 \tag{20}$$

From $(18)_1$, (19), and (20) we deduce that $\pi_0 = \gamma P_1$ and that

$$P_1 = \int_\lambda^1 \frac{K(2t^2 + K)\phi_{10}(t)}{\gamma t(t^2 + K)^2}\,dt \tag{21}$$

By considering the equilibrium of a sector cut off by the planes $\theta = \pm\theta_0$ we deduce that for the zero order contribution we require that

$$2\int_\lambda^1 c^2\sigma_{\theta\theta}\xi\,d\xi = P_1(\lambda^2 + K) \tag{22}$$

and from $(18)_2$ and (19), changing orders of integration, we can show that (22) becomes

$$\int_\lambda^1 \left\{\frac{\phi_{10}(\xi)}{\gamma^2}\left(\frac{\xi^2}{\xi^2 + K} + \frac{\xi^2 + K}{\xi^2}\right) - 2\gamma\phi_{20}(\xi)\right\}\xi\,d\xi = 0 \tag{23}$$

For a prescribed pressure $P_1$, (21) and (23) constitute two equations for the determination of the two constants $K$ and $\gamma$.

For example, for the Mooney material with strain–energy function (13), eqs. (21) and (23) yield, respectively,

$$\frac{P_1}{C_1} = \frac{1 + \gamma^2\Gamma}{\gamma}\left\{\frac{K(1 - \lambda^2)}{(\lambda^2 + K)(1 + K)} - 2\log\left[\lambda\left|\frac{1 + K}{\lambda^2 + K}\right|^{1/2}\right]\right\} \tag{24}$$

$$\frac{(\gamma^4 - \Gamma)}{\gamma(1 - \Gamma\gamma^2)} + \frac{K}{1 - \lambda^2}\log\left[\lambda\left|\frac{1 + K}{\lambda^2 + K}\right|^{1/2}\right] = 1 \tag{25}$$

In (16) and throughout this section we have tacitly assumed $(\xi^2 + K)$ is always positive so that the square root is meaningful. Strictly speaking, in order to include the possibility of $K$ large and negative, we should clearly interpret $(\xi^2 + K)^{1/2}$ as $|\xi^2 + K|^{1/2}$. This happens to be the case for the particular problem under consideration, and accordingly the modulus signs appear in (24) and (25). For this problem $\gamma > 1$, and, for prescribed $\lambda$, $\Gamma$, and $P_1/C_1$, (24) and (25) constitute two equations for the determination of $\gamma$ and $K$. The variation of $P_1/C_1$ with $\gamma$ for various values of $\Gamma$ is shown in Figure 3. From (24) and (25) directly it is apparent that for $\Gamma$ zero the pressure tends to zero with increasing $\gamma$. However, for $\Gamma$ nonzero the pressure ultimately increases with $\Gamma$. Figure 4 shows the variation of $K$ with $\gamma$ for $\Gamma$ zero and for $\Gamma = 0.2$. In general, $K$ is positive in the range $1 < \gamma < \Gamma^{-1/2}$ so that for the neo-Hookean material ($C_2 = 0$) $K$ is always positive. However, for $\Gamma$ nonzero, $K$ is a discontinuous function of $\gamma$ with $\gamma = \Gamma^{-1/2}$ as an asymptote, as indicated in Figure 4 for $\Gamma = 0.2$. We notice, however, that $P_1/C_1$ is a continuous function of $\gamma$.

## UNIFORM INFLATION TO A TORUS

In this section we consider the problem first studied by Kydoniefs and Spencer.[1] By introducing the same symbols as those employed in the previous section we can with slight changes make use of some of the equations already given. We

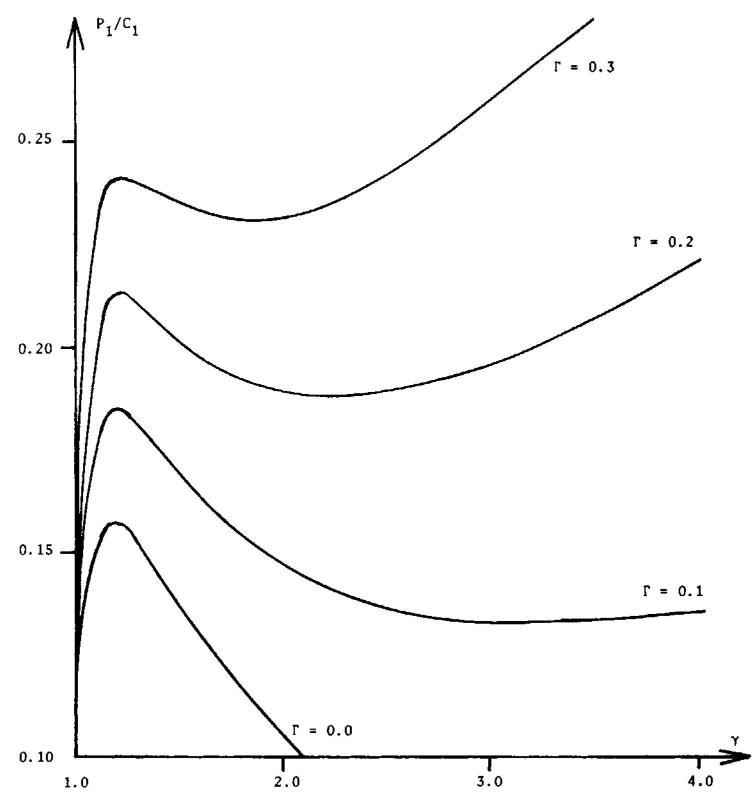


Fig. 3. Variation of $P_1/C_1$ with $\gamma$ for various values of $\Gamma$ and $\lambda = 0.9$.

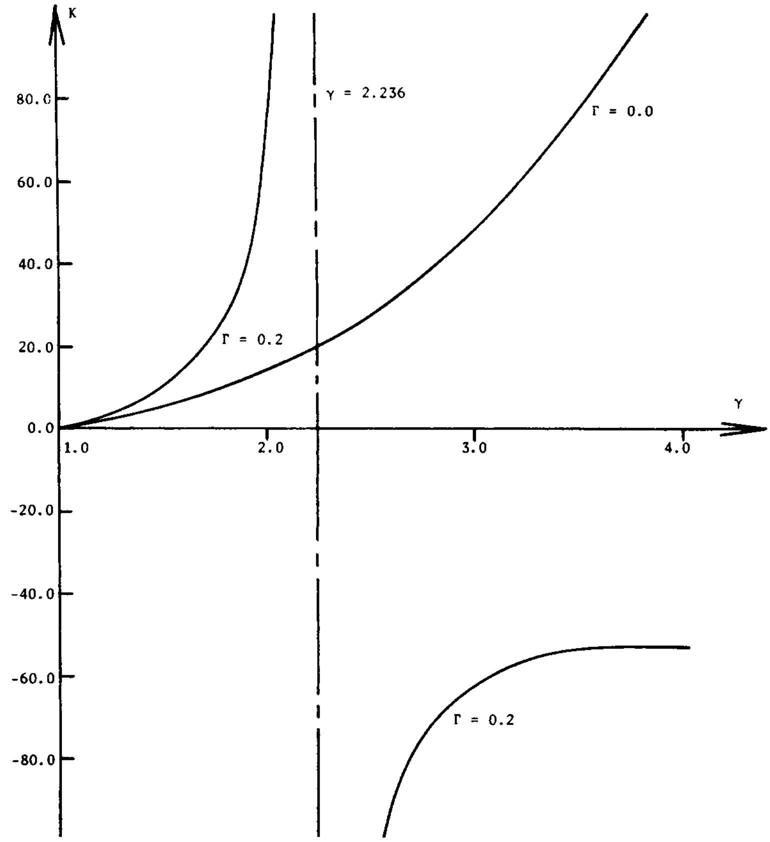


Fig. 4. Variation of $K$ with $\gamma$ for two values of $\Gamma$ and $\lambda = 0.9$.

suppose that a torus in its deformed state is obtained by rotating the region between two concentric circles of radii $r_1$ and $r_2$ about the $z$ axis and is such that $\epsilon = r_2/c$ is small compared with unity. We introduce

$$\xi = \frac{r}{r_2}, \quad \eta = \frac{R}{r_2}, \quad \lambda = \frac{r_1}{r_2}, \quad \gamma = \frac{b}{c} \tag{26}$$

so that again we have $\lambda \leqslant \xi \leqslant 1$. The deformation (8) becomes

$$\eta = \eta(\xi,\phi), \ \Theta = \theta, \quad \Phi = \Phi(\xi,\phi) \tag{27}$$

which we assume depends analytically on $\epsilon$ so that the zero-order contribution is again that given by (16). The zero-order contributions to $J$ and $\beta$ are given by $(17)_1$ and $(17)_2$, respectively, while we find that the physical components of the stress tensor become

$$\begin{aligned}
\sigma_{rr} &= \frac{1}{\gamma}\left\{\bar{p}_0(\xi) + \gamma\Sigma(J_0,\beta_0) - \frac{\xi^2}{\xi^2 + K}\,\psi_{10}(\xi)\right\} + O(\epsilon) \\
c^2\,\sigma_{\theta\theta} &= \frac{1}{\gamma}\left\{\bar{p}_0(\xi) + \gamma\Sigma(J_0,\beta_0) - \gamma^3\psi_{20}(\xi)\right\} + O(\epsilon) \\
\sigma_{\phi\phi} &= \frac{1}{\gamma}\left\{\bar{p}_0(\xi) + \gamma\Sigma(J_0,\beta_0) - \frac{\xi^2 + K}{\xi^2}\,\psi_{10}(\xi)\right\} + O(\epsilon)
\end{aligned} \tag{28}$$

$$\sigma_{r\phi} = O(\epsilon)$$

and all other components are zero. In (28) the pressure function $\bar{p}_0(\xi)$ is given by (19) with $\psi_{10}(\xi)$ in place of $\phi_{10}(\xi)$. For an applied internal pressure $P_2$ and zero external pressure, we deduce from boundary conditions of the form (20) that $\pi_0 = -\gamma[P_2 + \Sigma_0(\lambda)]$ and that

$$P_2 = -\int_\lambda^1 \frac{K(2t^2+K)\psi_{10}(t)}{\gamma t^3(t^2+K)}\,dt \tag{29}$$

Again by considering the equilibrium of a sector cut off by the planes $\theta = \pm\theta_0$ we require

$$2\int_\lambda^1 c^2\sigma_{\theta\theta}\xi d\xi = P_2\lambda^2 \tag{30}$$

On using the result,

$$\bar{p}_0(\xi) + \gamma\Sigma_0(\xi) = \frac{\xi^2}{\xi^2+K}\psi_{10}(\xi) + \int_\xi^1 \frac{K(2t^2+K)\psi_{10}(t)}{t^3(t^2+K)}\,dt \tag{31}$$

and interchanging orders of integration, we obtain from $(28)_2$ and (30)

$$\int_\lambda^1 \left\{\frac{\psi_{10}(\xi)}{\gamma^2}\left(\frac{\xi^2}{\xi^2+K} + \frac{\xi^2+K}{\xi^2}\right) - 2\gamma\psi_{20}(\xi)\right\}\xi d\xi = 0 \tag{32}$$

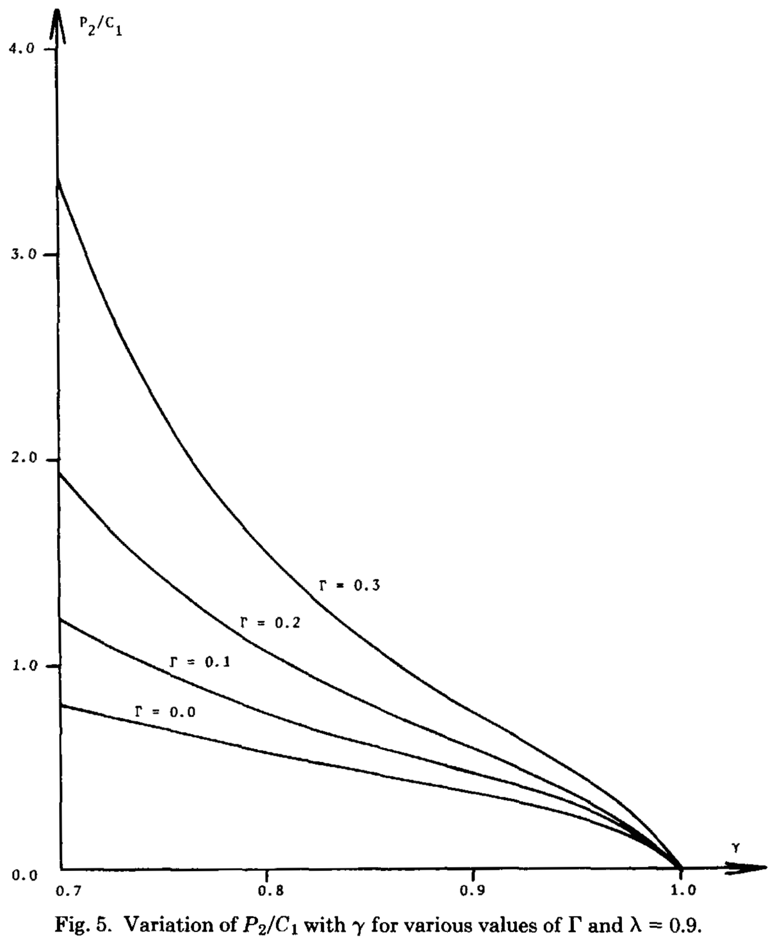


Fig. 5. Variation of $P_2/C_1$ with $\gamma$ for various values of $\Gamma$ and $\lambda = 0.9$.

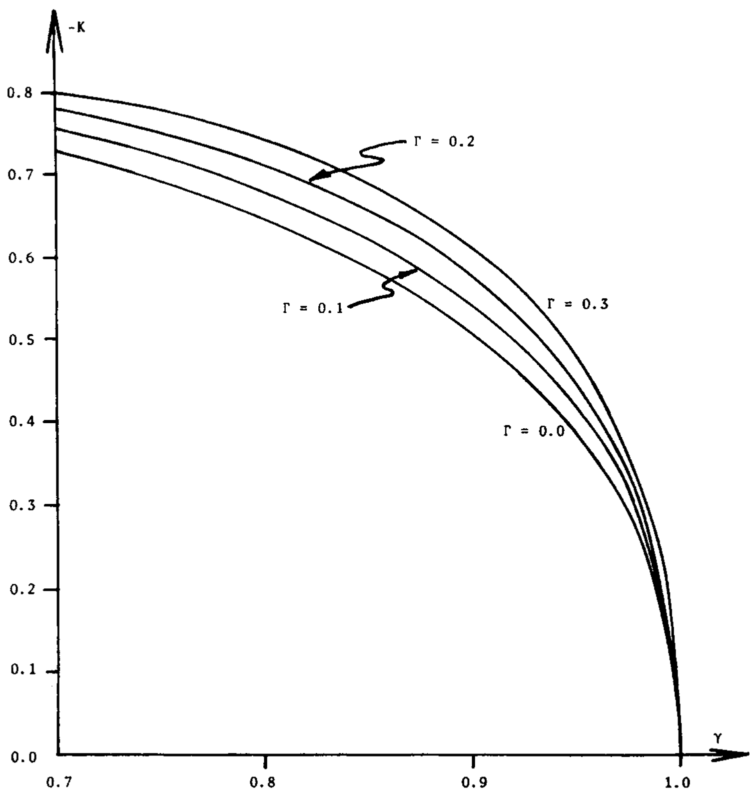


Fig. 6. Variation of $-K$ with $\gamma$ for various values of $\Gamma$ and $\lambda = 0.9$.

Thus, for a prescribed pressure $P_2$, (29) and (32) constitute two equations for the determination of $K$ and $\gamma$, and we observe that (32) takes the same form as (23), with $\psi_{10}$ and $\psi_{20}$ in place of $\phi_{10}$ and $\phi_{20}$, respectively.

For the Mooney material with strain–energy function (13) eqs. (29) and (32) become, respectively,

$$\frac{P_2}{C_1} = \frac{\gamma^2 + \Gamma}{\gamma}\left\{\frac{K(\lambda^2 - 1)}{\lambda^2} + 2 \log\left[\lambda\left(\frac{1 + K}{\lambda^2 + K}\right)^{1/2}\right]\right\} \tag{33}$$

$$\frac{(\Gamma\gamma^4 - 1)}{\gamma(\Gamma - \gamma^2)} + \frac{K}{(1 - \lambda^2)} \log\left[\lambda\left(\frac{1 + K}{\lambda^2 + K}\right)^{1/2}\right] = 1 \tag{34}$$

For this problem $\gamma < 1$ and the variation of $P_2/C_1$ with $\gamma$ is illustrated in Figure 5 for various values of $\Gamma$. The variation of $-K$ with $\gamma$ is shown in Figure 6. We observe that, as $\gamma$ decreases, $-K$ asymptotes $\lambda^2$ and that $-\lambda^2 < K < 0$ so that modulus signs do not appear in (33) and (34). We also note that in this case the results given here are in agreement with those given by Kydoniefs and Spencer[1] for the special case of the neo-Hookean material.

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